# Face-Turning Octahedron

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## 1 Generating the inequalities

If a face-turning octahedron can be built with $n$ layers physically with all parts anchored into the core, it can be modeled as a unit sphere with cuts placed at certain depths. Letting axes $a^1 = \frac{1}{\sqrt{3}}(-1,1,1)$, $a^2 = \frac{1}{\sqrt{3}}(1,-1,1)$, $a^3 = \frac{1}{\sqrt{3}}(1,1,-1)$, and $a^4 = \frac{1}{\sqrt{3}}(1,1,1)$, the puzzle can be made by placing the cuts at depths $d_{-n+2}, d_{-n+4}, \ldots, d_{n-4}, d_{n-2}$ along each of these axes. We can call the layers on axis $a^i$ between these cuts $L^i_{-n+1}, L^i_{-n+3}, \ldots, L^i_{n-3}, L^i_{n-1}$. We must have that

$$d_{i+1} < d_{i-1}. \tag{D$_i$}$$

To ensure symmetry, we have that $d_{-i} = -d_i$. This implies that $d_0 = 0$ when $n$ is even and $d_1 > 0$ when $n$ is odd (collectively the M inequality). We define the matrix $A = \begin{bmatrix} a^1 & a^2 & a^3 \end{bmatrix}$.

In order for this system of cuts to define a valid face-turning octahedron, possibly with extra piece types, every part expected on a face-turning octahedron, defined by the intersections of four layers from different axes, must intersect the surface of our sphere. There are four kinds of parts of a face-turning octahedron: the corners, the edges, and two kinds of centers. Without loss of generality, I will only be considering representative pieces from each piece orbit.

The corners are the intersection of $L^1_{-n+1}$, $L^2_{n-1}$, $L^3_{n-1}$, and $L^4_{n-1}$. The region is shaped like an infinite square cone with vertex $(\sqrt{3}d_{n-2}, 0, 0)$, and so for it to intersect the sphere, we must have that

$$d_{n-2} < \gamma \tag{V}$$

where $\gamma = \frac{1}{\sqrt{3}}$.

The edges are the intersection of $L^1_{-i}$, $L^2_i$, $L^3_{n-1}$, and $L^4_{n-1}$ for $-n+3 \leq i \leq n-3$. If inequality V is satisfied, the point

$$\frac{1}{2}(\sqrt{3}d + \sqrt{2-3d^2}, -\sqrt{3}d + \sqrt{2-3d^2}, 0),$$

where $d = \frac{d_{i+1}+d_{i-1}}{2}$, is in the intersection and on the surface of the sphere. Thus, no new inequalities need to be added to the system.

The centers are the intersection of $L_i^1$, $L_j^2$, $L_k^3$, and $L_{n-1}^4$. They come in two types: tetrahedral centers have $i+j+k=n+1$ and $-n+1 \leq i,j,k \leq n-1$, and octahedral centers have $i+j+k=n-1$ and $-n+3 \leq i,j,k \leq n-3$. Geometrically, the two types of centers differ by the angle of the cuts that bound them: tetrahedral centers are defined by cuts that slant at an acute angle going into the body of the puzzle, and octahedral centers have cuts that slant outward. These give them their tetrahedral and octahedral shapes. The intersection of these four layers in general can form many shapes, but because of the V inequality, we can ignore the layer on axis 4. The intersection of the remaining three layers is a rhombohedron whose inner and outer vertices are at

$$A^{-1}\begin{bmatrix} d_{i\pm 1} \\ d_{j\pm 1} \\ d_{k\pm 1} \end{bmatrix}$$

respectively. In order for this to intersect the sphere, the inner vertex must be inside the sphere (or otherwise behind it), and the outer vertex must be outside the sphere, resulting in the inequalities

$$\begin{bmatrix} d_{i-1} & d_{j-1} & d_{k-1} \end{bmatrix} (AA^T)^{-1} \begin{bmatrix} d_{i-1} \\ d_{j-1} \\ d_{k-1} \end{bmatrix} < 1 \vee (a^1+a^2+a^3) \cdot \begin{bmatrix} d_{i-1} \\ d_{j-1} \\ d_{k-1} \end{bmatrix} < 0 \qquad (\mathrm{FI}_{i-1,j-1,k-1})$$

and

$$\begin{bmatrix} d_{i+1} & d_{j+1} & d_{k+1} \end{bmatrix} (AA^T)^{-1} \begin{bmatrix} d_{i+1} \\ d_{j+1} \\ d_{k+1} \end{bmatrix} > 1 \qquad (\mathrm{FO}_{i+1,j+1,k+1})$$

When one of $i,j,k = n-1$, the $\mathrm{FO}_{i+1,j+1,k+1}$ inequality is not necessary. The second condition in the FI family is necessary because without it, some especially large pieces would not satisfy the inequalities. The set of inequalities can also be reduced in size. Firstly, $i$, $j$, $k$ are symmetric. Secondly, if $\mathrm{FI}_{i+1,j,k}$, $\mathrm{FI}_{i,j+1,k}$, $\mathrm{FI}_{i,j,k+1}$, and $\mathrm{FI}_{i,j,k}$ all exist, then the final inequality is satisfied when the other three are, and it is redundant, and likewise with $\mathrm{FO}_{i-1,j,k}$, $\mathrm{FO}_{i,j-1,k}$, $\mathrm{FO}_{i,j,k-1}$, and $\mathrm{FO}_{i,j,k}$.

## 2 Solving the inequalities

Solving the inequalities can be accomplished with a divide-and-conquer algorithm. There are $\lceil \frac{n}{2} \rceil - 1$ unknowns, so we can start the search algorithm on the domain

$$D_0 = [0,\gamma]^{\lceil \frac{n}{2} \rceil - 1}$$

which ensures the V and M inequalities will be satisfied. Starting on a larger domain is also fine given that we check V and M explicitly.

Given an axis-aligned cuboidal domain $D$, to check if there is a solution to the inequalities, we first check every inequality individually. The D inequalities define half-spaces, so it suffices to check the vertices for a solution. The

PO inequalities define the exterior of a solid ellipsoidal cylinder, which when intersected with $D_0$ becomes axis-convex (any axis-aligned line segment with endpoints in the region is completely contained within the region), and thus it also suffices to check only the vertices of $D$ for a solution. Finally, the PI inequalities define the union of the interior of said ellipsoidal cylinder and a half-space, so it suffices to check the vertices of $D$ for a solution to each term individually.

If any inequality is satisfied on none of the vertices of $D$, it must have no solution in $D$ either, and so the system has no solutions. If one vertex of $D$ satisfies all inequalities, then a solution has been found. Otherwise, $D$ is split into $2^{\lceil \frac{n}{2} \rceil - 1}$ smaller boxes, and the search algorithm is repeated on all of them.

If a face-turning octahedron of order $n$ exists, by removing every other cut, it should produce a valid face-turning octahedron of order $\lceil \frac{n}{2} \rceil$. This is one possible way to show that there is a maximum order of face-turning octahedron.

# 3 Generalization

The procedure used to find the limit of the face-turning octahedra can also be used on other types of puzzles. With $\phi$ the golden ratio, replacing $a^1 = \frac{(\phi,-1,0)}{\sqrt{\phi^2+1}}$, $a^2$ and $a^3$ by cyclic permutations of $a^1$, and $\gamma = \frac{1}{\sqrt{5}}$, the procedure can be applied to the Icosamate series. Replacing $a^1 = (1,0,0)$, $a^2 = (0,1,0)$, $a^3 = (0,0,1)$, and $\gamma = \frac{1}{\sqrt{2}}$, the procedure can be applied to edge-turning tetrahedra (the Mastermorphynx series), although viewing them as equivalent to face-turning cubes shows that the highest order is $n = 3$.

I believe a similar approach could be used to show that there is a limit in this sense for the Pyraminx, corner-turning octahedron, and shallow corner-turning icosahedron series.

# 4 Results

The program is at `https://github.com/milojacquet/octlimit-haskell`. I ran the program for every octahedron up to order 18, and it returned cut depths for all octahedra of order 13 or less, which are shown in the following table. Orders 14 through 18 were impossible. These are not necessarily the simplest sets of cut depths, just the first ones found by the program. For conciseness, the numerators are provided in the table and their common denominator is shown separately.

| n | $d_0, d_1$ | $d_2, d_3$ | $d_4, d_5$ | $d_6, d_7$ | $d_8, d_9$ | $d_{10}, d_{11}$ | denom. |
|---|---|---|---|---|---|---|---|
| 3 | 1 | | | | | | 2 |
| 4 | 0 | 1 | | | | | 2 |
| 5 | 1 | 2 | | | | | 4 |
| 6 | 0 | 1 | 2 | | | | 4 |
| 7 | 751 | 3072 | 4096 | | | | 8192 |
| 8 | 0 | 4 | 6 | 9 | | | 16 |
| 9 | 3 | 4 | 8 | 9 | | | 16 |
| 10 | 0 | 13 | 24 | 32 | 36 | | 64 |
| 11 | 271 | 512 | 906 | 1024 | 1152 | | 2048 |
| 12 | 0 | 789 | 1463 | 1944 | 2218 | 2364 | 4096 |
| 13 | 6079 | 18624 | 27444 | 33733 | 36564 | 37837 | 65536 |

For the icosahedron, I was only able to run the program up to order 16. The results are in the table below, in the same format.

| n | $d_0, d_1$ | $d_2, d_3$ | $d_4, d_5$ | $d_6, d_7$ | $d_8, d_9$ | $d_{10}, d_{11}$ | $d_{12}, d_{13}$ | $d_{14}$ | denom. |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | | | | | | | | 4 |
| 4 | 0 | 1 | | | | | | | 4 |
| 5 | 1 | 2 | | | | | | | 8 |
| 6 | 0 | 1 | 3 | | | | | | 8 |
| 7 | 1 | 2 | 3 | | | | | | 8 |
| 8 | 0 | 1 | 2 | 3 | | | | | 8 |
| 9 | 1 | 3 | 4 | 6 | | | | | 16 |
| 10 | 0 | 3 | 7 | 9 | 12 | | | | 32 |
| 11 | 4 | 8 | 15 | 20 | 24 | | | | 64 |
| 12 | 0 | 1 | 3 | 4 | 5 | 7 | | | 16 |
| 13 | 2 | 3 | 7 | 8 | 12 | 13 | | | 32 |
| 14 | 0 | 1 | 2 | 4 | 5 | 6 | 7 | | 16 |
| 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | 16 |
| 16 | 0 | 2 | 4 | 7 | 8 | 10 | 12 | 14 | 32 |